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ANALYSIS OF DUFFING'S OSCILLATOR EQUATION WITH TIME-DEPENDENT PARAMETERS

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In this study, an undamped Duffing's oscillator equation with time-dependent parameters has been considered. The time-varying part is expanded in a series of ultraspherical polynomials in the spirit of Sinha and Chou and only the constant part is retained. The non-linearity parameter is assumed to be small so that the number of iterations required is only two. The results compare well with those obtained by the Runge–Kutta fourth order method.

1. INTRODUCTION

In this study an undamped Duffing's oscillator with time-dependent parameters has been considered. The time-varying part is expanded in ultraspherical polynomials as in Sinha and Chou, and only the constant term is retained. The resulting equation is solved using the perturbation method [1]. The non-linearity parameter is assumed to be small, so that only two iterations are enough. The results have been compared with those obtained by the Runge–Kutta fourth order numerical integration method, and the agreement between the two is good. The present work deals with periodic coefficients only.

2. ANALYSIS

Consider the following Duffing's oscillator equation with time-dependent parameters:

$$\ddot{x} + (a_1 + b_1 \cos \omega_1 t)x + (a_2 + b_2 \cos \omega_2 t)x^3 = F_1. \tag{1}$$

Let the complete time interval $[0, T]$ be divided into n subintervals such as $[0, T_1], [T_1, T_2], [T_2, T_3], \ldots, [T_{n-1}, T_n]$. In each subinterval $\bar{T}_k$, the coefficients $(a_1 + b_1 \cos \omega_1 t)$ and $(a_2 + b_2 \cos \omega_2 t)$ are approximated by constants A_{k0} and ε_{k0}, respectively, by ultraspherical polynomial expansion of $\cos \omega_1 t$ and $\cos \omega_2 t$ [2]:

$$A_{k0} = a_1 + b_1 \cos(b_k \omega_1) \Lambda_\lambda(\omega_1 a_k),$$

$$\varepsilon_{k0} = a_2 + b_2 \cos(b_k \omega_2) \Lambda_\lambda(\omega_2 a_k), \tag{2}$$

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where a_k, b_k and $\Lambda_\lambda(\omega a_k)$ are given by

$$a_k = (T_k - T_{k-1})/2, \qquad b_k = (T_k + T_{k-1})/2, \tag{3, 4}$$

$$\Lambda_\lambda(\omega a_k) = \Gamma(\lambda + 1)\, J_\lambda(\omega a_k)/(\omega a_k/2)^\lambda, \tag{5}$$

$$\Gamma(\lambda + 1) = \int_0^\infty \bar{e}^t t^\lambda \, \mathrm{d}t$$

and

$$J_\lambda(\omega a_k) = \sum_{n=1}^{\infty} (-1)^n \frac{(\omega a_k/2)^{\lambda + 2n}}{(n!\,\Gamma(\lambda + n + 1))}.$$

Thus, in subinterval $\bar{T}_k$, equation (1) becomes

$$\ddot{x}_k + A_{k0} x_k + \varepsilon_{k0} x_k^3 = F_1. \tag{6}$$

If $\varepsilon_{k0} = 0$,

$$x_k = a \cos A_{k0}^{1/2} t + b \sin A_{k0}^{1/2} t + \frac{F_1}{A_{k0}}$$

is the solution, where a and b are obtained from initial conditions.

If ε_{k0} is not zero, but small, x_k can be assumed as

$$x_k = a \cos A_{k01}^{1/2} t + b \sin A_{k01}^{1/2} t + \frac{F_1}{A_{k01}}, \tag{7}$$

where A_{k01} differs a little from A_{k0}, and a and b are obtained from the initial conditions $x_k \dot{x}_k$ at the start time T_{k-1} of the interval $\bar{T}_k$ and are given as

$$b = \left[x_k(T_{k-1}) - \frac{F_1}{A_{k01}} \right] \sin \sqrt{A_{k01}}\, T_{k-1} + \frac{1}{\sqrt{A_{k01}}} [\dot{x}_k(T_{k-1})] \times \cos \sqrt{A_{k01}}\, T_{k-1}, \tag{8}$$

$$a = \left[x_k(T_{k-1}) - \frac{F_1}{A_{k01}} - b \sin \sqrt{A_{k01}}\, T_{k-1} \right] \frac{1}{\cos \sqrt{A_{k01}}\, T_{k-1}}. \tag{9}$$

By writing A_{k0} as

$$A_{k0} = A_{k01} + (A_{k0} - A_{k01}), \tag{10}$$

equation (6) becomes

$$\ddot{x}_k + A_{k01} x_k + (A_{k0} - A_{k01}) x_k + \varepsilon_{k0} x_k^3 = F_1. \tag{11}$$

Substituting equation (7) into equation (11) yields

$$\ddot{x}_k + A_{k01} x_k = -(A_{k0} - A_{k01}) \left(a \cos A_{k01}^{1/2} t + b \sin A_{k01}^{1/2} t + \frac{F_1}{A_{k01}} \right)$$

$$- \varepsilon_{k0} \left(a \cos A_{k01}^{1/2} t + b \sin A_{k01}^{1/2} t + \frac{F_1}{A_{k01}} \right)^3 + F_1. \tag{12}$$

Simplifying the above equation, we get

$$\ddot{x}_k + A_{k01} x_k = z_1 \cos 3A_{k01}^{1/2} t + z_2 \sin 3A_{k01}^{1/2} t + z_3 \cos 2A_{k01}^{1/2} t + z_4 \sin 2A_{k01}^{1/2} t + z_5 \cos A_{k01}^{1/2} t + z_6 \sin A_{k01}^{1/2} t + \text{Force} - \text{term}, \tag{13}$$

where

$$z_1 = -\varepsilon_{k0}\left(\frac{a^3}{4} - \frac{3b^2 a}{4}\right),$$

$$z_2 = -\varepsilon_{k0}\left(-\frac{b^3}{4} + \frac{3a^2 b}{4}\right),$$

$$z_3 = -\varepsilon_{k0}\left(\frac{3a^2 F_1}{2A_{k01}} - \frac{3b^2 F_1}{2A_{k01}}\right),$$

$$z_4 = -\varepsilon_{k0}\left(\frac{3abF_1}{A_{k01}}\right),$$

$$z_5 = -\varepsilon_{k0}\left(\frac{3a^3}{4} + \frac{3b^2 a}{4} + \frac{3aF_1^2}{A_{k01}^2} + \frac{(A_{k0} - A_{k01})a}{\varepsilon_{k0}}\right),$$

$$z_6 = -\varepsilon_{k0}\left(\frac{3b^3}{4} + \frac{3a^2 b}{4} + \frac{3bF_1^2}{A_{k01}^2} + \frac{(A_{k0} - A_{k01})b}{\varepsilon_{k0}}\right),$$

$$\text{Force} - \text{term} = -\left(\frac{(A_{k0} - A_{k01})}{A_{k01}}\right)F_1 + F_1 - \frac{3\varepsilon_{k0}}{2A_{k01}} a^2 F_1 - \frac{3\varepsilon_{k0}}{2A_{k01}} b^2 F_1 - \frac{\varepsilon_{k0}}{A_{k01}^3} F_1^3. \tag{14}$$

In order to avoid resonance, the coefficients of $\cos A_{k01}^{1/2} t$ and $\sin A_{k01}^{1/2} t$ are to be equated to zero, i.e, $z_5 = z_6 = 0$. This gives the relation

$$A_{k01} = A_{k0} + \varepsilon_{k0}\left(\frac{3}{4}\right)(a^2 + b^2) + \frac{3\varepsilon_{k0} F_1^2}{A_{k0}^2}. \tag{15}$$

Then the solution to equation (13) is given by

$$x_k = c_1 \cos A_{k01}^{1/2} t + c_2 \sin A_{k01}^{1/2} t + \left(-\frac{z_1}{8A_{k01}}\right)\cos 3A_{k01}^{1/2} t + \left(-\frac{z_2}{8A_{k01}}\right)\sin 3A_{k01}^{1/2} t + \left(-\frac{z_3}{3A_{k01}}\right)\cos 2A_{k01}^{1/2} t + \left(-\frac{z_4}{3A_{k01}}\right)\sin 2A_{k01}^{1/2} t + \frac{\text{Force} - \text{term}}{A_{k01}}. \tag{16}$$

Thus the solution for the interval $\bar{T}_k$ is obtained. The solution for the complete time interval $[0, T]$ is obtained using the solutions of successive intervals.

TABLE 1

Results of the solution of Duffing's oscillator equation with time-dependent parameters

Sr. number	A_{k0}	ε_{k0}	a	b	c_1	c_2	$\bar{T}_k$
1	0·7380	0·2409	1·000	0·000	0·992	0·000	0, 0·6
	1·0721	0·1887	0·977	0·062	0·948	0·134	0·6, 1·2
	1·3142	0·1509	0·893	0·193	0·859	0·242	1·2, 1·8
	1·1556	0·1756	0·970	0·099	0·919	0·175	1·8, 2·4
2	0·7380	0·2409	0·323	0·000	0·411	0·000	0, 0·6
	1·0721	0·1887	0·499	0·107	0·513	0·157	0·6, 1·2
	1·3142	0·1509	0·486	0·244	0·465	0·286	1·2, 1·8
	1·1556	0·1756	0·576	0·143	0·538	0·220	1·8, 2·4
3	0·7380	0·2409	1·000	0·000	0·993	0·000	0, 0·6
	1·0721	0·1887	0·972	0·081	0·941	0·156	0·6, 1·2
	1·3142	0·1509	0·889	0·210	0·842	0·273	1·2, 1·8
	1·1556	0·1756	0·969	0·115	0·909	0·202	1·8, 2·4
4	0·7380	0·2409	0·323	0·000	0·399	0·000	0, 0·6
	1·0721	0·1887	0·495	0·118	0·511	0·170	0·6, 1·2
	1·3142	0·1509	0·485	0·252	0·455	0·307	1·2, 1·8
	1·1556	0·1756	0·579	0·152	0·535	0·239	1·8, 2·4

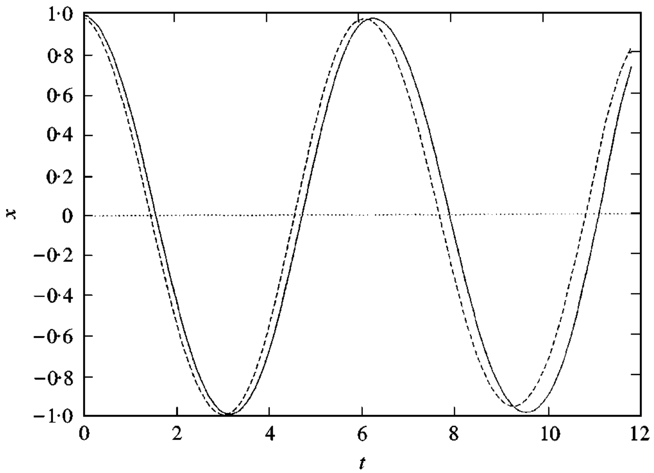

Figure 1. Displacement (x) versus time (t) of $\ddot{x} + (1 - 0{\cdot}32\cos 2t)x + (0{\cdot}2 + 0{\cdot}05\cos 2t)x^3 = 0$. Runge-Kutta method – – –; Sinha's method ——.

Table 1 gives some of the details of the systems studied. Figures 1–5 show the results for the following systems:

$$(1)\ \ddot{x} + (1 - 0{\cdot}32\cos 2t)x + (0{\cdot}2 + 0{\cdot}05\cos 2t)x^3 = 0,$$

$$x = 1, \quad \dot{x} = 0 \quad \text{at } t = 0.$$

$$(2)\ \ddot{x} + (1 - 0{\cdot}32\cos 2t)x + (0{\cdot}2 + 0{\cdot}05\cos 2t)x^3 = 0{\cdot}5,$$

$$x = 1, \quad \dot{x} = 0 \quad \text{at } t = 0$$

$$(3)\ \ddot{x} + (1 - 0{\cdot}32\cos 2t)x + 0{\cdot}2x^3 = 0,$$

$$x = 1, \quad \dot{x} = 0 \quad \text{at } t = 0.$$

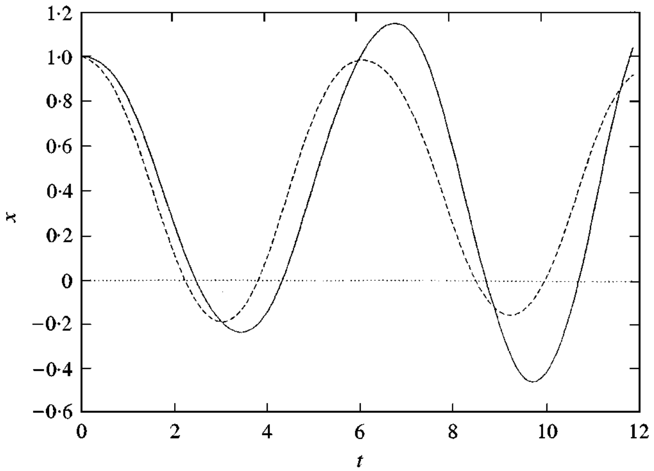

Figure 2. Displacement (x) versus time (t) of $\ddot{x} + (1 - 0{\cdot}32\cos 2t)x + (0{\cdot}2 + 0{\cdot}05\cos 2t)x^3 = 0{\cdot}5$. Runge-Kutta method – – –; Sinha's method ——.

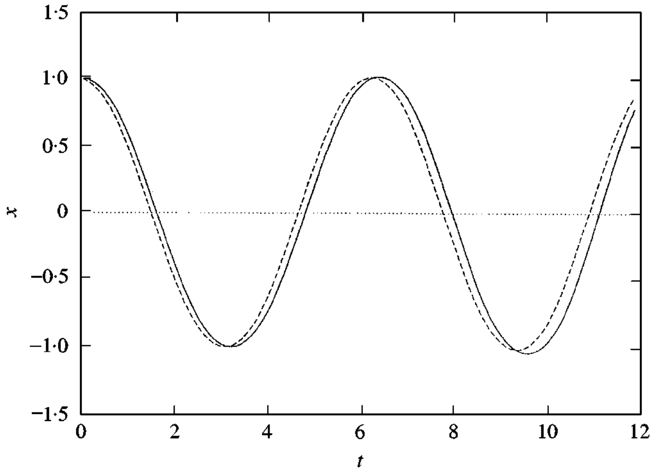

Figure 3. Displacement (x) versus time (t) of $\ddot{x} + (1 - 0{\cdot}32\cos 2t)x + 0{\cdot}2x^3 = 0$. Runge-Kutta method – – –; Sinha's method ——.

(4) $\ddot{x} + (1 - 0{\cdot}32\cos 2t)x + 0{\cdot}2x^3 = 0{\cdot}5,$

$$x = 1, \quad \dot{x} = 0 \quad \text{at } t = 0.$$

(5) $\ddot{x} + x + 0{\cdot}2x^3 = 0{\cdot}5,$

$$x = 1, \quad \dot{x} = 0 \quad \text{at } t = 0.$$

3. RESULTS AND DISCUSSION

In the study, the method proposed by Sinha and Chou [2] has been combined with the method of perturbation to obtain the response of Duffing's oscillator with time-varying parameters. The results obtained by the approximate method and the Runge–Kutta fourth order method are in good agreement. The value of λ can be chosen between 1 and 1000, but

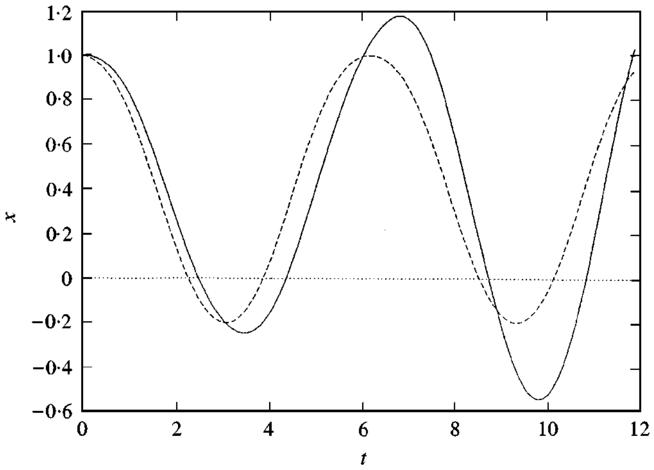

Figure 4. Displacement (x) versus time (t) of $\ddot{x} + (1 - 0{\cdot}32\cos 2t)x + 0{\cdot}2x^3 = 0{\cdot}5$. Runge-Kutta method ---; Sinha's method ——.

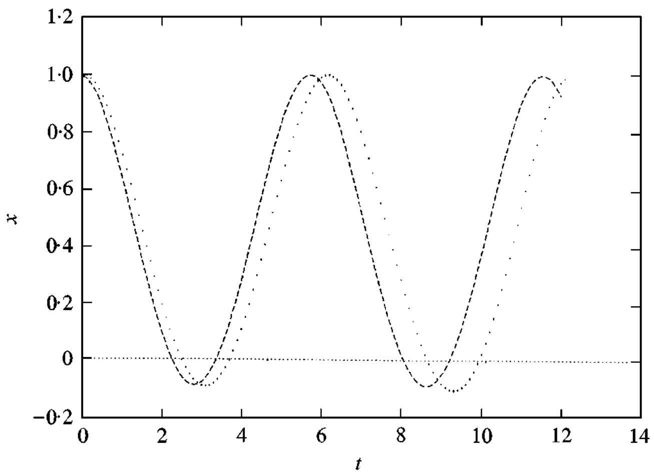

Figure 5. Displacement (x) versus time (t) of $\ddot{x} + x + 0{\cdot}2x^3 = 0{\cdot}5$. Runge-Kutta method ---; Sinha's method ..... .

there is no criterion for choosing the optimum value of λ. The accuracy of the solution can be improved by reducing the interval size and adjusting the value of λ. The results for $F_1 = 0$, shown in Figures 1 and 3, are in better agreement than those for $F_1 = 0{\cdot}5$, shown in Figures 2 and 4. In Figure 5 results agree well where $F_1 = 0{\cdot}5$. Hence, it can be concluded that the errors in Figures 2 and 4 are due to the constant approximation.

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